High order in time schemes for hyperbolic conservation laws by an approximate Cauchy-Kovalevskaya procedure

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In this work we present a numerical method to solve with arbitrarily high order in space and time hyperbolic conservation laws of the form

\[ u_t + \nabla \cdot f(u) = 0, \quad u(x, 0) = u_0(x), \]

where \( u = u(x, t) \colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^m \), \( f : \mathbb{R}^m \to \mathbb{R}^m \).

High order of accuracy can be obtained for example by using time-advance semi-discrete methods, where the divergence is first approximated by means of techniques based on flux reconstructions like the WENO method [1] for example, leading to ODE systems for time that are then solved by means of methods that provide high order in time, like the TVD Runge-Kutta schemes of Shu and Osher [3].

Another approach was introduced in [2] and consists on a Taylor expansion on time of the unknown, so that the scheme

\[ u^{n+1} = u^n + \sum_{l=1}^R \frac{\tilde{u}^{(l)}(x, t)}{k!} \Delta t^l \]

is \( R \)-th order accurate if \( \tilde{u}^{(l)}(x, t) = u^{(l)}(x, t) + O(\Delta t^{R+1-l}) \).

By using the equation, the time derivatives of \( u \) can be computed from space derivatives, in turn obtained by finite-difference approximations. For example, for a scalar equation in one dimension one has \( u_t = -f(u)_x \), \( u_{tt} = (f'(u)^2 u_x)_x \) for the first two derivatives. In the general case the expressions involved in the time derivatives can become very complex, requiring the use of symbolic calculus software and resulting on computationally expensive algorithms.

The main contribution of this work is an alternative version of the previous method [4] that replaces the exact expressions of the time derivatives of the fluxes by approximations of the adequate order, obtained through finite-difference formulas. Hence, the resulting scheme does not require exact flux derivatives and keeps the accuracy order, leading to a much simpler implementation. In cases where the flux derivatives are complex the approximate version obtains better performance than the exact version. The combination of such schemes with known high order spatial reconstructions, provides a high order method in both space and time in a straightforward way.

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References


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